

# ON THE CURVATURE ESTIMATES FOR HESSIAN EQUATIONS

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ABSTRACT. The curvature estimates of  $k$  curvature equations for general right hand side is a longstanding problem. In this paper, we totally solve the  $n - 1$  case and we also discuss some applications for our estimate.

## 1. INTRODUCTION

In this paper, we continue to study the longstanding problem of global  $C^2$  estimates for curvature equation in general type,

$$(1.1) \quad \sigma_k(\kappa(X)) = f(X, \nu(X)), \quad \forall X \in M,$$

where  $\sigma_k$  is the  $k$ th elementary symmetric function,  $\nu(X)$ ,  $\kappa(X)$  are the outer-normal and principal curvatures of hypersurface  $M \subset \mathbb{R}^{n+1}$  at the position vector  $X$  respectively.

Equation (1.1) is the general form of some important type equations. For the cases  $k = 1, 2$  and  $n$ , they are the mean curvature, scalar curvature and Gauss curvature type equation. We will mainly discuss the case of  $k = n - 1$  in this paper.

Now, let's give a brief review of some history related these equations. A lot of geometric problems fall into equation (1.1) with special form of  $f$ . The famous Minkowski problem, namely, prescribed Gauss-Kronecker curvature on the outer normal, has been widely discussed in [25, 26, 27, 12]. Alexandrov also posed the problem of prescribing general Weingarten curvature on outer normals, seeing [2, 18]. The prescribing curvature measures problem in convex geometry also has been extensively studied in [1, 26, 20, 19]. In [3, 30, 10], the prescribing mean curvature problem and Weingarten curvature problem also have been considered and obtained fruitful results.

In many case, the main difficulty of the equation (1.1) is trying to obtain  $C^2$  estimates. Hence, let's review some known results. For  $k = 1$ , equation (1.1) is quasilinear,  $C^2$  estimate follows from the classical theory of quasilinear PDE. The equation is of Monge-Ampère type if  $k = n$ .  $C^2$  estimate in this case for general  $f(X, \nu)$  is due to Caffarelli-Nirenberg-Spruck [8]. When  $f$  is independent of normal vector  $\nu$ ,  $C^2$  estimate has been proved by Caffarelli-Nirenberg-Spruck [10]. If  $f$  in (1.1) depends only on  $\nu$ ,  $C^2$  estimate was proved in [18]. Ivochkina [22, 23] considered the Dirichlet problem of equation (1.1) on domains in  $\mathbb{R}^n$ ,  $C^2$  estimate was proved there under some extra conditions on the dependence of  $f$  on  $\nu$ .  $C^2$  estimate was also proved for equation of prescribing curvature measures problem in [20, 19], where  $f(X, \nu) = \langle X, \nu \rangle \tilde{f}(X)$ . For  $k = 2$  and convex case, the  $C^2$  estimate

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Research of the last author is supported by an NSFC Grant No.11301087.

have been obtained in [21]. Recently, the scalar curvature case is generalized and simplified in [29]. For general equation (1.1), the desired  $C^2$  estimate should be in the Garding cone  $\Gamma_k$ . Following [9], the Garding's cone is defined by,

**Definition 1.** For a domain  $\Omega \subset \mathbb{R}^n$ , a function  $v \in C^2(\Omega)$  is called  $k$ -convex if the eigenvalues  $\kappa(x) = (\kappa_1(x), \dots, \kappa_n(x))$  of the hessian  $\nabla^2 v(x)$  is in  $\Gamma_k$  for all  $x \in \Omega$ , where  $\Gamma_k$  is the Garding's cone

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \mid \sigma_m(\kappa) > 0, \quad m = 1, \dots, k\}.$$

A  $C^2$  regular hypersurface  $M \subset \mathbb{R}^{n+1}$  is  $k$ -convex if  $\kappa(X) \in \Gamma_k$  for all  $X \in M$ .

In the present paper, for  $n - 1$  Hessian equation, we can obtain the  $C^2$  estimate in  $\Gamma_{n-1}$ . Namely, totally solve the  $C^2$  estimate for  $n - 1$  Hessian equation. In fact, the main result of this paper is,

**Theorem 2.** Suppose  $M \subset \mathbb{R}^{n+1}$  is a closed  $n - 1$ -convex hypersurface satisfying curvature equation (1.1) with  $k = n - 1$  for some positive function  $f(X, \nu) \in C^2(\Gamma)$ , where  $\Gamma$  is an open neighborhood of unit normal bundle of  $M$  in  $\mathbb{R}^{n+1} \times \mathbb{S}^n$ , then there is a constant  $C$  depending only on  $n, k, \|M\|_{C^1}, \inf f$  and  $\|f\|_{C^2}$ , such that

$$(1.2) \quad \max_{X \in M, i=1, \dots, n} \kappa_i(X) \leq C.$$

We use two steps to prove the above estimate. The first key step is to obtain a better inequality which we have got in section 2. This is more explicit estimate than the inequalities obtained in [21]. Then using the test function discovered in [21], we obtain the global  $C^2$  estimate.

We also have the similar estimate for Dirichlet problem in  $\mathbb{R}^n$ .

**Corollary 3.** For the Dirichlet problem of  $\sigma_{n-1}$  equation defined in some bounded domain  $\Omega \subset \mathbb{R}^n$ , it is,

$$(1.3) \quad \begin{cases} \sigma_{n-1}[D^2u] &= f(x, u, Du), & \text{in } \Omega \\ u &= \varphi, & \text{on } \partial\Omega \end{cases}$$

The global  $C^2$  estimates can be obtained. It means that, we have some constants  $C$  depending on  $f$  and  $\nabla u, u$  and the domain  $\Omega$ , such that,

$$\|u\|_{C^2(\bar{\Omega})} \leq C + \max_{\partial\Omega} |\nabla^2 u|.$$

More reference about these type of estimates can be found in [13], [24] and therein.

Now, let's exhibit some applications of our estimate. The first application is that we can obtain the corresponding existence result for  $n - 1$ -convex solutions of the prescribed  $n - 1$  curvature equation (1.1). For the sake of the  $C^0, C^1$  estimates, we need further barrier conditions on the prescribed function  $f$  as considered in [3, 30, 10]. We denote  $\rho(X) = |X|$ .

We assume that

*Condition (1).* There are two positive constant  $r_1 < 1 < r_2$  such that

$$(1.4) \quad \begin{cases} f(X, \frac{X}{|X|}) \geq \frac{\sigma_k(1, \dots, 1)}{r_1^k}, & \text{for } |X| = r_1, \\ f(X, \frac{X}{|X|}) \leq \frac{\sigma_k(1, \dots, 1)}{r_2^k}, & \text{for } |X| = r_2. \end{cases}$$

*Condition (2).* For any fixed unit vector  $\nu$ ,

$$(1.5) \quad \frac{\partial}{\partial \rho}(\rho^k f(X, \nu)) \leq 0, \quad \text{where } |X| = \rho.$$

Using the above two condition, we have the following existence theorem.

**Theorem 4.** Suppose  $k = n - 1$  and suppose positive function  $f \in C^2(\bar{B}_{r_2} \setminus B_{r_1} \times \mathbb{S}^n)$  satisfies conditions (1.4) and (1.5), then equation (1.1) has a unique  $C^{3,\alpha}$  starshaped solution  $M$  in  $\{r_1 \leq |X| \leq r_2\}$ .

We also can apply our estimate to the prescribed curvature problem for spacelike graph hypersurface in Minkowski space. We assume the graph can be written by function  $u$  which means that  $(x, u(x)), x \in \mathbb{R}^n$  is its position vector. Still, we suppose  $\kappa_1, \dots, \kappa_n$  be the principal curvature of these hypersurface. The principal curvature can be written by the derivative of the function  $u$  which will be more clear in section 4. We have the following theorem.

**Theorem 5.** Let  $\Omega$  be some bounded domain in  $\mathbb{R}^n$  with smooth boundary and  $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$  is a positive function with  $f_u \geq 0$ . Let  $\varphi \in C^4(\bar{\Omega})$  be space like. Consider the following Dirichlet problem,

$$(1.6) \quad \begin{cases} \sigma_{n-1}(\kappa_1, \dots, \kappa_n) = f(x, u, Du), & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega. \end{cases}$$

If the above problem have some sub soultion, then it has a unique space like solution  $u$  in  $\Gamma_{n-1}$  belonging to  $C^{3,\alpha}(\bar{\Omega})$  for any  $\alpha \in (0, 1)$ .

The prescribed curvature problem for spacelike graph hypersurface in Minkowski space is proposed by Bayard [6, 7]. The scalar curvature case has been totally solved by Urban [31]. The above theorem solves  $k = n - 1$  case. For the rest case  $2 < k < n - 1$ , it is still open. The difference with the problem in Euclidean space is that the curvature term has opposite sign. Hence, even for function  $f$  does not depend on gradient term, these problem can not be successful solved as in Euclidean space, comparing [11]. Hypersurfaces of prescribed curvature problem in Lorentzian manifolds also have been extensively studied by Bartnik-Simon [5], Delanoë [14], Gerhardt [15, 16] and Schnürer [28].

In this paper, we use standard notation. We let  $\kappa(A)$  be eigenvalues of the matrix  $A = (a_{ij})$ . For equation

$$F(A) = F(\kappa(A)),$$

we define

$$F^{pq} = \frac{\partial F}{\partial a_{pq}}, \text{ and } F^{pq,rs} = \frac{\partial^2 F}{\partial a_{pq} \partial a_{rs}}.$$

For a local orthonormal frame, if  $A$  is diagonal at a point, then at this point,

$$F^{pp} = \frac{\partial f}{\partial \kappa_p} = f_p, \text{ and } F^{pp,qq} = \frac{\partial^2 f}{\partial \kappa_p \partial \kappa_q} = f_{pq}.$$

The following facts regarding  $\sigma_k$  will be used throughout this paper.

- (i)  $\sigma_k^{pp,pp} = 0$  and  $\sigma_k^{pp,qq}(\kappa) = \sigma_{k-2}(\kappa|pq)$ ;
- (ii)  $\sigma_k^{pq,rs} h_{pql} h_{rsl} = \sigma_k^{pp,qq} h_{pql}^2 - \sigma_k^{pp,qq} h_{ppl} h_{qql}$ .

Here, the notation  $\sigma_l(\kappa|ab\cdots)$  means  $l$  symmetric function exclude the indices  $a, b, \cdots$ . Now, we give the following two Lemmas, which will be needed in our proof.

**Lemma 6.** Set  $k > l$ . For  $\alpha = \frac{1}{k-l}$ , we have,

$$(1.7) \quad \begin{aligned} & -\frac{\sigma_k^{pp,qq}}{\sigma_k} u_{pph} u_{qqh} + \frac{\sigma_l^{pp,qq}}{\sigma_l} u_{pph} u_{qqh} \\ & \geq \left( \frac{(\sigma_k)_h}{\sigma_k} - \frac{(\sigma_l)_h}{\sigma_l} \right) \left( (\alpha-1) \frac{(\sigma_k)_h}{\sigma_k} - (\alpha+1) \frac{(\sigma_l)_h}{\sigma_l} \right). \end{aligned}$$

further more, for sufficiently small  $\delta > 0$ , we have,

$$(1.8) \quad \begin{aligned} & -\sigma_k^{pp,qq} u_{pph} u_{qqh} + (1 - \alpha + \frac{\alpha}{\delta}) \frac{(\sigma_k)_h^2}{\sigma_k} \\ & \geq \sigma_k (\alpha + 1 - \delta\alpha) \left[ \frac{(\sigma_l)_h}{\sigma_l} \right]^2 - \frac{\sigma_k}{\sigma_l} \sigma_l^{pp,qq} u_{pph} u_{qqh}. \end{aligned}$$

The another one is,

**Lemma 7.** Denote  $Sym(n)$  the set of all  $n \times n$  symmetric matrices. Let  $F$  be a  $C^2$  symmetric function defined in some open subset  $\Psi \subset Sym(n)$ . At any diagonal matrix  $A \in \Psi$  with distinct eigenvalues, let  $\ddot{F}(B, B)$  be the second derivative of  $C^2$  symmetric function  $F$  in direction  $B \in Sym(n)$ , then

$$(1.9) \quad \ddot{F}(B, B) = \sum_{j,k=1}^n \ddot{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j < k} \frac{\dot{f}^j - \dot{f}^k}{\kappa_j - \kappa_k} B_{jk}^2.$$

The proof of the first Lemma can be found in [19] and [21]. The second Lemma can be found in [4] and [9].

The organization of the paper is as follow. We give the key inequality in section 2. Theorem 2 is proved in section 3. in section 4, we obtain some applications.

## 2. AN INEQUALITY

In this section, we will prove the following Proposition. It is a explicit inequality. We consider the  $\sigma_{n-1}$  equation in  $n$  dimensional space.

**Proposition 8.** For any index  $i$  and  $\varepsilon$ , if  $\kappa_i \geq \delta \kappa_1$ , then we have,

$$(2.1) \quad \kappa_i [K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq} u_{ppi} u_{qqi}] - \sigma_{n-1}^{ii} u_{iii}^2 + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj} u_{jji}^2 \geq 0.$$

for sufficient large  $K$  depending on  $\delta$  and  $\varepsilon$ .

*Proof.* A directly calculation shows,

$$\begin{aligned}
(2.2) \quad & \kappa_i[K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq}u_{ppi}u_{qqi}] - \sigma_{n-1}^{ii}u_{iii}^2 + (1+\varepsilon)\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}^2 \\
&= \kappa_i K[\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}]^2 + 2\kappa_i u_{iii}[\sum_{j \neq i} (K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj} - \sigma_{n-1}^{ii,jj})u_{jji}] \\
&\quad + (\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii})u_{iii}^2 + (1+\varepsilon)\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}^2 - \kappa_i \sum_{p \neq i; q \neq i} \sigma_{n-1}^{pp,qq}u_{ppi}u_{qqi} \\
&\geq \kappa_i K[\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}]^2 - \frac{\kappa_i^2[\sum_{j \neq i} (K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj} - \sigma_{n-1}^{ii,jj})u_{jji}]^2}{\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii}} \\
&\quad + (1+\varepsilon)\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}^2 - \kappa_i \sum_{p \neq i; q \neq i} \sigma_{n-1}^{pp,qq}u_{ppi}u_{qqi} \\
&= \sum_{j \neq i} [\kappa_i K(\sigma_{n-1}^{jj})^2 - \frac{\kappa_i^2(K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj} - \sigma_{n-1}^{ii,jj})^2}{\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii}} + (1+\varepsilon)\sigma_{n-1}^{jj}]u_{jji}^2 \\
&\quad + \sum_{p, q \neq i; p \neq q} [\kappa_i K\sigma_{n-1}^{pp}\sigma_{n-1}^{qq} - \frac{\kappa_i^2(K\sigma_{n-1}^{ii}\sigma_{n-1}^{pp} - \sigma_{n-1}^{ii,pp})(K\sigma_{n-1}^{ii}\sigma_{n-1}^{qq} - \sigma_{n-1}^{ii,qq})}{\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii}} \\
&\quad - \kappa_i \sigma_{n-1}^{pp,qq}]u_{ppi}u_{qqi},
\end{aligned}$$

where, in the second inequality, we have used,

$$\begin{aligned}
& \frac{\kappa_i^2[\sum_{j \neq i} (K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj} - \sigma_{n-1}^{ii,jj})u_{jji}]^2}{\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii}} + 2\kappa_i u_{iii}[\sum_{j \neq i} (K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj} - \sigma_{n-1}^{ii,jj})u_{jji}] \\
& + (\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii})u_{iii}^2 \geq 0.
\end{aligned}$$

Note that we have,

$$K\kappa_i\sigma_{n-1}^{ii} - 1 \geq K\delta\kappa_1\sigma_{n-1}^{11} - 1 \geq 0,$$

for sufficient large  $K$ . Hence, we can omit the denominator in (2.2). Then, we get,

$$\begin{aligned}
(2.3) \quad & (\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii})[\kappa_i[K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq}u_{ppi}u_{qqi}] - \sigma_{n-1}^{ii}u_{iii}^2 \\
& + (1+\varepsilon)\sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}^2] \\
&\geq \sum_{j \neq i} [\kappa_i K\sigma_{n-1}^{ii}\sigma_{n-1}^{jj}(-\sigma_{n-1}^{jj} + 2\kappa_i\sigma_{n-1}^{ii,jj} + (1+\varepsilon)\sigma_{n-1}^{ii}) - \kappa_i^2(\sigma_{n-1}^{ii,jj})^2 \\
& - (1+\varepsilon)\sigma_{n-1}^{ii}\sigma_{n-1}^{jj}]u_{jji}^2 \\
& + \sum_{p, q \neq i; p \neq q} [\kappa_i K\sigma_{n-1}^{ii}(\kappa_i(\sigma_{n-1}^{pp}\sigma_{n-1}^{ii,qq} + \sigma_{n-1}^{qq}\sigma_{n-1}^{ii,pp} - \sigma_{n-1}^{ii}\sigma_{n-1}^{pp,qq}) - \sigma_{n-1}^{pp}\sigma_{n-1}^{qq}) \\
& - \kappa_i^2\sigma_{n-1}^{ii,pp}\sigma_{n-1}^{ii,qq} + \kappa_i\sigma_{n-1}^{ii}\sigma_{n-1}^{pp,qq}]u_{ppi}u_{qqi}.
\end{aligned}$$

We have several identities. At first, we have,

$$-\sigma_{n-1}^{jj} + 2\kappa_i\sigma_{n-1}^{ii,jj} + \sigma_{n-1}^{ii} = (\kappa_i + \kappa_j)\sigma_{n-3}(\kappa|i,j).$$

Hence, we get,

$$\begin{aligned}
(2.4) \quad & \sigma_{n-1}^{jj}(\kappa_i + \kappa_j)\sigma_{n-3}(\kappa|ij) \\
&= (\kappa_i\sigma_{n-2}(\kappa|j) + \sigma_{n-1} - \sigma_{n-1}(\kappa|j))\sigma_{n-3}(\kappa|ij) \\
&= (\kappa_i(\kappa_i\sigma_{n-3}(\kappa|ij) + \sigma_{n-2}(\kappa|ij)) - \kappa_i\sigma_{n-2}(\kappa|ij) + \sigma_{n-1})\sigma_{n-3}(\kappa|ij) \\
&= \kappa_i^2(\sigma_{n-3}(\kappa|ij))^2 + \sigma_{n-1}\sigma_{n-3}(\kappa|ij),
\end{aligned}$$

where we have used  $\sigma_{n-1}(\kappa|ij) = 0$ . We also have,

$$\begin{aligned}
(2.5) \quad & \kappa_i(\sigma_{n-1}^{pp}\sigma_{n-1}^{ii,qq} + \sigma_{n-1}^{qq}\sigma_{n-1}^{ii,pp} - \sigma_{n-1}^{ii}\sigma_{n-1}^{pp,qq}) - \sigma_{n-1}^{pp}\sigma_{n-1}^{qq} \\
&= \kappa_i\sigma_{n-1}^{qq}\sigma_{n-1}^{ii,pp} - \kappa_i\sigma_{n-1}^{ii}\sigma_{n-1}^{pp,qq} - \sigma_{n-1}^{pp}\sigma_{n-2}(\kappa|iq) \\
&= \kappa_i\sigma_{n-2}(\kappa|pq)\sigma_{n-3}(\kappa|ip) - \kappa_i\sigma_{n-2}(\kappa|ip)\sigma_{n-3}(\kappa|pq) - \sigma_{n-2}(\kappa|p)\sigma_{n-2}(\kappa|iq) \\
&= (\kappa_i^2\sigma_{n-3}(\kappa|ipq) + \kappa_i\sigma_{n-2}(\kappa|ipq))(\kappa_q\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq)) \\
&\quad - (\kappa_q\sigma_{n-3}(\kappa|ipq) + \sigma_{n-2}(\kappa|ipq))(\kappa_i^2\sigma_{n-4}(\kappa|ipq) + \kappa_i\sigma_{n-3}(\kappa|ipq)) \\
&\quad - \sigma_{n-2}(\kappa|p)(\kappa_p\sigma_{n-3}(\kappa|ipq) + \sigma_{n-2}(\kappa|ipq)) \\
&= \kappa_i^2(\sigma_{n-3}(\kappa|ipq))^2 - \kappa_i\kappa_q(\sigma_{n-3}(\kappa|ipq))^2 - \kappa_p\sigma_{n-2}(\kappa|p)\sigma_{n-3}(\kappa|ipq) \\
&= \kappa_i^2(\sigma_{n-3}(\kappa|ipq))^2 - \sigma_{n-1}\sigma_{n-3}(\kappa|ipq).
\end{aligned}$$

Here we have used  $\sigma_{n-2}(\kappa|ipq) = 0$  and

$$\sigma_{n-1} = \kappa_p\sigma_{n-2}(\kappa|p) + \sigma_{n-1}(\kappa|p) = \kappa_p\sigma_{n-2}(\kappa|p) + \kappa_i\kappa_q\sigma_{n-3}(\kappa|ipq).$$

We also have,

$$\begin{aligned}
(2.6) \quad & \sigma_{n-1}^{ii,pp}\sigma_{n-1}^{ii,qq} \\
&= (\kappa_q\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq))(\kappa_p\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq)) \\
&= (\sigma_{n-3}(\kappa|ipq))^2 + [\kappa_p\kappa_q\sigma_{n-4}(\kappa|ipq) + (\kappa_p + \kappa_q)\sigma_{n-3}(\kappa|ipq)]\sigma_{n-4}(\kappa|ipq) \\
&= (\sigma_{n-3}(\kappa|ipq))^2 + \sigma_{n-2}(\kappa|i)\sigma_{n-4}(\kappa|ipq),
\end{aligned}$$

where we have used

$$\sigma_{n-2}(\kappa|i) = \kappa_p\sigma_{n-3}(\kappa|ip) + \sigma_{n-2}(\kappa|ip) = \kappa_p\kappa_q\sigma_{n-4}(\kappa|ipq) + (\kappa_p + \kappa_q)\sigma_{n-3}(\kappa|ipq).$$

We have,

$$(2.7) \quad \sigma_{n-1}^{pp,qq} = \kappa_i\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq)$$

Using the above two identities (2.6) and (2.7), we get,

$$\begin{aligned}
(2.8) \quad & -\kappa_i^2\sigma_{n-1}^{ii,pp}\sigma_{n-1}^{ii,qq} + \kappa_i\sigma_{n-1}^{ii}\sigma_{n-1}^{pp,qq} \\
&= -\kappa_i^2(\sigma_{n-3}(\kappa|ipq))^2 + \kappa_i\sigma_{n-1}^{ii}\sigma_{n-3}(\kappa|ipq).
\end{aligned}$$

Using identities (2.4), (2.5) and (2.8), (2.3) becomes,

$$\begin{aligned}
& (\kappa_i K(\sigma_{n-1}^{ii})^2 - \sigma_{n-1}^{ii})[\kappa_i[K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq}u_{ppi}u_{qqi}] - \sigma_{n-1}^{ii}u_{iii}^2 \\
& + (1 + \varepsilon) \sum_{j \neq i} \sigma_{n-1}^{jj}u_{jji}^2] \\
\geq & \sum_{j \neq i} [\kappa_i K \sigma_{n-1}^{ii} (\kappa_i^2 (\sigma_{n-3}(\kappa|ij))^2 + \sigma_{n-3}(\kappa|ij)\sigma_{n-1} + \varepsilon \sigma_{n-1}^{jj} \sigma_{n-1}^{ii}) \\
& - \kappa_i^2 (\sigma_{n-1}^{ii,jj})^2 - (1 + \varepsilon) \sigma_{n-1}^{ii} \sigma_{n-1}^{jj}] u_{jji}^2 \\
& + \sum_{p,q \neq i, p \neq q} [\kappa_i K \sigma_{n-1}^{ii} (\kappa_i^2 (\sigma_{n-3}(\kappa|ipq))^2 - \sigma_{n-1} \sigma_{n-3}(\kappa|ipq)) \\
& - \kappa_i^2 (\sigma_{n-3}(\kappa|ipq))^2 + \kappa_i \sigma_{n-1}^{ii} \sigma_{n-3}(\kappa|ipq)] u_{ppi} u_{qqi} \\
= & \sum_{j \neq i} [(\kappa_i K \sigma_{n-1}^{ii} - 1) \kappa_i^2 (\sigma_{n-3}(\kappa|ij))^2 + \kappa_i K \sigma_{n-1}^{ii} \sigma_{n-3}(\kappa|ij) \sigma_{n-1} \\
& + (\kappa_i K \sigma_{n-1}^{ii} \varepsilon - (1 + \varepsilon)) \sigma_{n-1}^{ii} \sigma_{n-1}^{jj}] u_{jji}^2 \\
& + \sum_{p,q \neq i, p \neq q} [(\kappa_i K \sigma_{n-1}^{ii} - 1) \kappa_i^2 (\sigma_{n-3}(\kappa|ipq))^2 \\
& - \kappa_i K \sigma_{n-1}^{ii} (\sigma_{n-1} - \frac{1}{K}) \sigma_{n-3}(\kappa|ipq)] u_{ppi} u_{qqi} \\
\geq & \sum_{j \neq i} [(\kappa_i K \sigma_{n-1}^{ii} - 1) \kappa_i^2 (\sigma_{n-3}(\kappa|ij))^2 + \kappa_i K \sigma_{n-1}^{ii} \sigma_{n-3}(\kappa|ij) (\sigma_{n-1} - \frac{1}{K})] u_{jji}^2 \\
& + \sum_{p,q \neq i, p \neq q} [(\kappa_i K \sigma_{n-1}^{ii} - 1) \kappa_i^2 (\sigma_{n-3}(\kappa|ipq))^2 \\
& - \kappa_i K \sigma_{n-1}^{ii} (\sigma_{n-1} - \frac{1}{K}) \sigma_{n-3}(\kappa|ipq)] u_{ppi} u_{qqi}.
\end{aligned}$$

Here, the last inequality holds for sufficient large  $K$ . Now, we only need to check whether the following two bilinear form are nonnegative. There are

$$(2.9) \quad \sum_{j \neq i} (\sigma_{n-3}(\kappa|ij))^2 u_{jji}^2 + \sum_{p,q \neq i, p \neq q} (\sigma_{n-3}(\kappa|ipq))^2 u_{ppi} u_{qqi},$$

and,

$$(2.10) \quad \sum_{j \neq i} \sigma_{n-3}(\kappa|ij) u_{jji}^2 - \sum_{p,q \neq i, p \neq q} \sigma_{n-3}(\kappa|ipq) u_{ppi} u_{qqi}.$$

Let's consider the corresponding two matrices. Denote

$$a_{pq} = \begin{cases} \sigma_{n-3}(\kappa|ip), & p = q \\ -\sigma_{n-3}(\kappa|ipq), & p \neq q \end{cases}.$$

Now we need a elemental theorem in linear algebra. That is the Schur product theorem for Hadmard product.

**Theorem 9.** *The Hadmard product of two semipositive definite matrices is semipositive definite.*

Here, the meaning of the Hadamard product is that every entry of the product of two matrices is the directly product of corresponding entries of two matrices. For example, if matrices  $B = (b_{ij}), C = (c_{ij})$ , then the Hadamard product of matrices  $B, C$  is the matrix  $(b_{ij}c_{ij})$ . Thus, to prove the bilinear forms of (2.9) and (2.10) are semi positive forms, we only need to prove the matrices  $(a_{pq})$  and  $(a_{pq}^2)$  are semi positive definite. By Schur's product theorem, we only need to check that the matrix  $(a_{pq})$  is semi positive definite. It comes from the following Lemma.  $\square$

**Lemma 10.** *Suppose  $2 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n$  are  $m$  ordered indices. Then,  $D_m(i_1, \dots, i_m)$  the  $k$ -th principal sub determinant of the matrix  $(a_{pq})$  is*

$$(2.11) \quad D_m(i_1, \dots, i_m) = \det \begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_m} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_m} \\ \dots & \dots & \dots & \dots \\ a_{i_m i_1} & a_{i_m i_2} & \dots & a_{i_m i_m} \end{bmatrix} \\ = \sigma_{n-2}^{m-1}(\kappa|1)\sigma_{n-(m+2)}(\kappa|1i_1 \dots i_m).$$

The another needed determinant is, for  $k \neq m$ ,

$$(2.12) \quad B_{m-1}(i_1, \dots, i_m; i_k) \\ = \det \begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_{k-1}} & a_{i_1 i_{k+1}} & \dots & a_{i_1 i_m} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_{k-1}} & a_{i_2 i_{k+1}} & \dots & a_{i_2 i_m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i_{m-1} i_1} & a_{i_{m-1} i_2} & \dots & a_{i_{m-1} i_{k-1}} & a_{i_{m-1} i_{k+1}} & \dots & a_{i_{m-1} i_m} \end{bmatrix} \\ = (-1)^{m+k} [\sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \dots i_{k-1} i_{k+1} \dots i_{m-1}) \\ + \sigma_{n-m}(\kappa|1i_2 \dots i_m) \sigma_{n-2}^{m-3}(\kappa|1) \sum_{l \neq k, m} \sigma_{n-3}(\kappa|1i_1 i_l)].$$

Hence, in  $\Gamma_{n-1}$  cone, we have,

$$D_{m-1}(2 \dots m) = \sigma_{n-2}^{m-2}(\kappa|1)\sigma_{n-(m+1)}(\kappa|12 \dots m) > 0,$$

which implies the matrix  $(a_{pq})$  is a nonnegative definite matrix.

*Proof.* We prove the above two formulas by induction.

For  $m = 2$ ,

$$B_1(i_1 i_2; i_1) = a_{i_1 i_2} = -\sigma_{n-3}(\kappa|1i_1 i_2).$$

Also, we have, by (2.6),

$$D_2(i_1 i_2) = \sigma_{n-3}(\kappa|1i_1)\sigma_{n-3}(\kappa|1i_2) - \sigma_{n-3}^2(\kappa|1i_1 i_2) \\ = \sigma_{n-2}(\kappa|1)\sigma_{n-4}(\kappa|1i_1 i_2).$$



Hence, we assume that (2.11) and (2.12) both hold for less than  $m - 1$ . For  $m$  case, we have,

$$\begin{aligned}
(2.13) \quad & D_m(i_1 \cdots, i_m) \\
&= \sum_{l=1}^{m-1} (-1)^{m+l} a_{i_m i_l} B_{m-1}(i_1 \cdots i_m; i_l) + a_{i_m i_m} D_{m-1}(i_1 \cdots i_{m-1}) \\
&= \sigma_{n-2}^{m-3}(\kappa|1) [\sigma_{n-2}(\kappa|1) \sigma_{n-3}(\kappa|1 i_m) \sigma_{n-(m+1)}(\kappa|1 i_1 \cdots i_{m-1}) \\
&\quad - \sum_{l=1}^{m-1} \sigma_{n-3}^2(\kappa|1 i_l i_m) \sigma_{n-m}(\kappa|1 i_1 \cdots i_{l-1} i_{l+1} \cdots i_{m-1}) \\
&\quad - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1 i_l i_m) \sigma_{n-m}(\kappa|1 i_2 \cdots i_m) \sum_{k \neq l, m} \sigma_{n-3}(\kappa|1 i_1 i_k)].
\end{aligned}$$

We also have,

$$\begin{aligned}
(2.14) \quad & \sigma_{n-3}(\kappa|1 i_l i_m) \sigma_{n-m}(\kappa|1 i_1 \cdots i_{l-1} i_{l+1} \cdots i_{m-1}) \\
&\quad + \sigma_{n-m}(\kappa|1 i_2 \cdots i_m) \sum_{k \neq l, m} \sigma_{n-3}(\kappa|1 i_1 i_k) \\
&= \sigma_{n-3}(\kappa|1 i_l i_m) (\kappa_{i_l} \sigma_{n-m-1}(\kappa|1 i_1 \cdots i_{m-1}) + \sigma_{n-m}(\kappa|1 i_1 \cdots i_{m-1})) \\
&\quad + \sigma_{n-m}(\kappa|1 i_1 \cdots i_{m-1}) \sum_{k \neq l} \sigma_{n-3}(\kappa|1 i_k i_m) \\
&= \sigma_{n-2}(\kappa|1 i_m) \sigma_{n-m-1}(\kappa|1 i_1 \cdots i_{m-1}) + \sigma_{n-m}(\kappa|1 i_1 \cdots i_{m-1}) \sum_k \sigma_{n-3}(\kappa|1 i_k i_m) \\
&= \sigma_{n-3}(\kappa|1 i_1 i_m) (\kappa_{i_1} \sigma_{n-m-1}(\kappa|1 i_1 \cdots i_{m-1}) + \sigma_{n-m}(\kappa|1 i_1 \cdots i_{m-1})) \\
&\quad + \sigma_{n-m}(\kappa|1 i_1 \cdots i_{m-1}) \sum_{k \neq 1} \sigma_{n-3}(\kappa|1 i_k i_m) \\
&= \sigma_{n-3}(\kappa|1 i_1 i_m) \sigma_{n-m}(\kappa|1 i_2 \cdots i_{m-1}) \\
&\quad + \sigma_{n-m+1}(\kappa|1 i_2 \cdots i_{m-1}) \sum_k \sigma_{n-4}(\kappa|1 i_1 i_k i_m) \\
&= \sigma_{n-4}(\kappa|1 i_1 i_2 i_m) (\kappa_{i_2} \sigma_{n-m}(\kappa|1 i_2 \cdots i_{m-1}) + \sigma_{n-m+1}(\kappa|1 i_2 \cdots i_{m-1})) \\
&\quad + \sigma_{n-m+1}(\kappa|1 i_2 \cdots i_{m-1}) \sum_{k \neq 2} \sigma_{n-4}(\kappa|1 i_1 i_k i_m) \\
&= \sigma_{n-4}(\kappa|1 i_1 i_2 i_m) \sigma_{n-m+1}(\kappa|1 i_3 \cdots i_{m-1}) \\
&\quad + \sigma_{n-m+2}(\kappa|1 i_3 \cdots i_{m-1}) \sum_k \sigma_{n-5}(\kappa|1 i_1 i_2 i_k i_m) \\
&= \cdots = \sigma_{n-(m+1)}(\kappa|1 i_1 \cdots i_m) \sigma_{n-2}(\kappa|1).
\end{aligned}$$

Hence, we have,

$$\begin{aligned}
(2.15) \quad & D_m(i_1 \cdots, i_m) \\
&= \sigma_{n-2}^{m-2}(\kappa|1)[\sigma_{n-3}(\kappa|1i_m)\sigma_{n-(m+1)}(\kappa|1i_1 \cdots i_{m-1}) \\
&\quad - \sigma_{n-m-1}(\kappa|1i_1 \cdots i_m) \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1i_l i_m)] \\
&= \sigma_{n-2}^{m-2}(\kappa|1)[\sigma_{n-3}(\kappa|1i_m)\kappa_{i_m}\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m) \\
&\quad + \sigma_{n-m-1}(\kappa|1i_1 \cdots i_m)(\sigma_{n-3}(\kappa|1i_m) - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1i_l i_m))].
\end{aligned}$$

It is clear that we have,

$$\begin{aligned}
(2.16) \quad & \sigma_{n-3}(\kappa|1i_m) - \sum_{l=1}^{m-1} \sigma_{n-3}(\kappa|1i_l i_m) \\
&= \kappa_{i_1}\sigma_{n-4}(\kappa|1i_1 i_m) - \sum_{l=2}^{m-1} \sigma_{n-3}(\kappa|1i_l i_m) \\
&= \kappa_{i_1}[\sigma_{n-4}(\kappa|1i_1 i_m) - \sum_{l=2}^{m-1} \sigma_{n-4}(\kappa|1i_1 i_l i_m)] \\
&= \kappa_{i_1}[\kappa_{i_2}\sigma_{n-5}(\kappa|1i_1 i_2 i_m) - \sum_{l=3}^{m-1} \sigma_{n-4}(\kappa|1i_1 i_l i_m)] \\
&= \kappa_{i_1}\kappa_{i_2}[\sigma_{n-5}(\kappa|1i_1 i_2 i_m) - \sum_{l=3}^{m-1} \sigma_{n-5}(\kappa|1i_1 i_2 i_l i_m)] \\
&= \cdots = \kappa_{i_1}\kappa_{i_2} \cdots \kappa_{i_{m-1}}\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m).
\end{aligned}$$

Hence, we obtain,

$$\begin{aligned}
& D_m(i_1 \cdots, i_m) \\
&= \sigma_{n-2}^{m-2}(\kappa|1)[\sigma_{n-3}(\kappa|1i_m)\kappa_{i_m}\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m) \\
&\quad + \sigma_{n-m-1}(\kappa|1i_1 \cdots i_m)\kappa_{i_1}\kappa_{i_2} \cdots \kappa_{i_{m-1}}\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m)] \\
&= \sigma_{n-2}^{m-2}(\kappa|1)[\sigma_{n-3}(\kappa|1i_m)\kappa_{i_m} + \sigma_{n-2}(\kappa|1i_m)]\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m) \\
&= \sigma_{n-2}^{m-1}(\kappa|1)\sigma_{n-(m+2)}(\kappa|1i_1 \cdots i_m).
\end{aligned}$$

For the formula (2.12), we can rewrite it to be,

$$\begin{aligned}
(2.17) \quad & B_{m-1}(i_1, \cdots, i_m; i_k) \\
&= (-1)^{m+k}[\sigma_{n-3}(\kappa|1i_k i_m)D_{m-2}(i_1 \cdots i_{k-1} i_{k+1} \cdots i_{m-1}) \\
&\quad + \sigma_{n-m}(\kappa|1i_1 \cdots i_{m-1})\sigma_{n-2}^{m-3}(\kappa|1) \sum_{l \neq k} \sigma_{n-3}(\kappa|1i_l i_m)].
\end{aligned}$$

Now, let's expand its last row to prove it. In what following,  $\hat{i}$  means that index  $i$  does not appear. We have,

$$\begin{aligned}
(2.18) \quad & B_{m-1}(i_1 \cdots i_m; i_k) \\
&= \sum_{l>k} (-1)^{m+l} \sigma_{n-3}(\kappa|1i_l i_m) (-1)^{m-2-l} B_{m-2}(i_1 \cdots i_{l-1} i_{l+1} \cdots i_{m-1} i_l; i_k) \\
&\quad + \sum_{l<k} (-1)^{m+l} \sigma_{n-3}(\kappa|1i_l i_m) (-1)^{m-1-l} B_{m-2}(i_1 \cdots i_{l-1} i_{l+1} \cdots i_{m-1} i_l; i_k) \\
&\quad + (-1)^{m+k} \sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \cdots i_{k-1} i_{k+1} \cdots i_{m-1}) \\
&= (-1)^{m-1+k-1} \sum_{l>k} \sigma_{n-3}(\kappa|1i_l i_m) [\sigma_{n-3}(\kappa|1i_k i_l) D_{m-3}(i_1 \cdots \hat{i}_l \cdots \hat{i}_k \cdots i_{m-1}) \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sigma_{n-2}^{m-4}(\kappa|1) \sum_{a \neq k} \sigma_{n-3}(\kappa|1i_l i_a)] \\
&\quad + (-1)(-1)^{m-1+k} \sum_{l<k} \sigma_{n-3}(\kappa|1i_l i_m) [\sigma_{n-3}(\kappa|1i_k i_l) D_{m-3}(i_1 \cdots \hat{i}_k \cdots \hat{i}_l \cdots i_{m-1}) \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sigma_{n-2}^{m-4}(\kappa|1) \sum_{a \neq k} \sigma_{n-3}(\kappa|1i_l i_a)] \\
&\quad + (-1)^{m+k} \sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \cdots i_{k-1} i_{k+1} \cdots i_{m-1}) \\
&= (-1)^{m+k} \left\{ \sum_{l \neq k} \sigma_{n-3}(\kappa|1i_l i_m) [\sigma_{n-3}(\kappa|1i_k i_l) D_{m-3}(i_1 \cdots \hat{i}_l \cdots \hat{i}_k \cdots i_{m-1}) \right. \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sigma_{n-2}^{m-4}(\kappa|1) \sum_{a \neq k} \sigma_{n-3}(\kappa|1i_l i_a)] \\
&\quad + \sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \cdots i_{k-1} i_{k+1} \cdots i_{m-1}) \left. \right\} \\
&= (-1)^{m+k} \left\{ \sigma_{n-2}^{m-4}(\kappa|1) \sum_{l \neq k} \sigma_{n-3}(\kappa|1i_l i_m) \right. \\
&\quad \times [\sigma_{n-3}(\kappa|1i_k i_l) \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots \hat{i}_k \cdots i_{m-1}) \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sum_{a \neq k} \sigma_{n-3}(\kappa|1i_l i_a)] \\
&\quad + \sigma_{n-3}(\kappa|1i_k i_m) D_{m-2}(i_1 \cdots i_{k-1} i_{k+1} \cdots i_{m-1}) \left. \right\}.
\end{aligned}$$

We see that,

$$\begin{aligned}
(2.19) \quad & \sigma_{n-3}(\kappa|1i_k i_l) \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots \hat{i}_k \cdots i_{m-1}) \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sum_{a \neq k} \sigma_{n-3}(\kappa|1i_l i_a) \\
&= \sigma_{n-2}(\kappa|1i_l) \sigma_{n-m}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1 \cdots \hat{i}_l \cdots i_{m-1}) \sum_a \sigma_{n-3}(\kappa|1i_l i_a)
\end{aligned}$$

$$\begin{aligned}
&= \sigma_{n-3}(\kappa|1i_1i_l)[\kappa_{i_1}\sigma_{n-m}(\kappa|1i_1\cdots\hat{i}_l\cdots i_{m-1}) + \sigma_{n-(m-1)}(\kappa|1i_1\cdots\hat{i}_l\cdots i_{m-1})] \\
&\quad + \sigma_{n-(m-1)}(\kappa|1i_1\cdots\hat{i}_l\cdots i_{m-1}) \sum_{a \neq 1} \sigma_{n-3}(\kappa|1i_li_a) \\
&= \sigma_{n-3}(\kappa|1i_1i_l)\sigma_{n-(m-1)}(\kappa|1i_2\cdots\hat{i}_l\cdots i_{m-1}) \\
&\quad + \sigma_{n-m+2}(\kappa|1i_2\cdots\hat{i}_l\cdots i_{m-1}) \sum_a \sigma_{n-4}(\kappa|1i_1i_li_a) \\
&= \sigma_{n-4}(\kappa|1i_1i_2i_l)[\kappa_{i_2}\sigma_{n-m+1}(\kappa|1i_2\cdots\hat{i}_l\cdots i_{m-1}) + \sigma_{n-m+2}(\kappa|1i_2\cdots\hat{i}_l\cdots i_{m-1})] \\
&\quad + \sigma_{n-m+2}(\kappa|1i_2\cdots\hat{i}_l\cdots i_{m-1}) \sum_{a \neq 2} \sigma_{n-4}(\kappa|1i_1i_li_a) \\
&= \sigma_{n-4}(\kappa|1i_1i_2i_l)\sigma_{n-m+2}(\kappa|1i_3\cdots\hat{i}_l\cdots i_{m-1}) \\
&\quad + \sigma_{n-m+3}(\kappa|1i_3\cdots\hat{i}_l\cdots i_{m-1}) \sum_a \sigma_{n-5}(\kappa|1i_1i_2i_li_a) \\
&= \cdots = \sigma_{n-m}(\kappa|1i_1\cdots i_{m-1})\sigma_{n-2}(\kappa|1).
\end{aligned}$$

Hence, combining (2.18) and (2.19), we obtain (2.17).  $\square$

At last, we give a counter example. This example says that our inequality holds only for  $\sigma_{n-1}$ . We consider the  $\sigma_2$  in dimension 4. Suppose

$$\kappa_1 = 2t + \frac{1}{t}, \kappa_2 = 2t, \kappa_3 = 0, \text{ and } \kappa_4 = -t.$$

Then, a directly calculate gives,

$$\sigma_2^{11} = t, \sigma_2^{22} = t + \frac{1}{t}, \sigma_2^{33} = 3t + \frac{1}{t}, \sigma_2^{44} = 4t + \frac{1}{t}, \sigma_2 = 1.$$

Hence,  $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$  is in  $\Gamma_2$  cone for  $t > 0$ . Let's calculate the determinate of the matrix defined by the bilinear form (2.1) for  $i = 1$  case. It is equal to,

$$275K - 311 + \frac{12K - 12}{t^4} + \frac{96K - 100}{t^2} + (313K - 427)t^2 + (66K - 216)t^4 - 72Kt^6.$$

Obviously, it is not nonnegative for sufficient large  $t$ .

### 3. GLOBAL CURVATURE ESTIMATE

In this section, we consider the global  $C^2$ -estimates for the curvature equation of  $k = n - 1$ . At first, we need the following Lemma.

**Lemma 11.** *For any constant  $0 < \varepsilon_T < \frac{1}{2}$ , there exist another constant  $0 < \delta < \min\{\varepsilon_T/2, 1/200\}$ , which depends on  $\varepsilon_T$ , such that, if  $|\kappa_i| < \delta\kappa_1$ , we have,*

$$(3.1) \quad (1 + \varepsilon_T)e^{\kappa_l}\sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T)\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|l) \geq \frac{e^{\kappa_l}}{\kappa_1}\sigma_{k-1}(\kappa|i),$$

for sufficient large  $\kappa_1$ .

*Proof.* It is obvious that we have the following identity,

$$\sigma_{k-1}(\kappa|l) = \sigma_{k-1}(\kappa|i) + (\kappa_i - \kappa_l)\sigma_{k-2}(\kappa|il).$$

Multiplying  $\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}$  in both side of the above identity, we have,

$$(3.2) \quad e^{\kappa_l}\sigma_{k-2}(\kappa|il) + \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|l) = e^{\kappa_i}\sigma_{k-2}(\kappa|il) + \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|i).$$

We divide into four cases to discuss.

Case (i):  $\kappa_l \leq \kappa_i$ .

In this case, we have,

$$\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|i) = e^{\kappa_l}\frac{e^{\kappa_i - \kappa_l} - 1}{\kappa_i - \kappa_l}\sigma_{k-1}(\kappa|i) \geq e^{\kappa_l}\sigma_{k-1}(\kappa|i).$$

Hence, by (3.2), we get (3.1) for sufficient large  $\kappa_1$ .

Case (ii):  $0 < \kappa_l - \kappa_i \leq 1$ .

In this case, obviously, we have  $\kappa_i \geq \kappa_l - 1$ . By the mean value theorem, there exists some constant  $\kappa_i < \xi < \kappa_l$ . Then we have,

$$\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|i) = e^{\xi}\sigma_{k-1}(\kappa|i) \geq e^{\kappa_l - 1}\sigma_{k-1}(\kappa|i) \geq \frac{e^{\kappa_l}}{\kappa_1}\sigma_{k-1}(\kappa|i),$$

if  $\kappa_1$  is sufficient large. By (3.2), we get (3.1).

Case(iii):  $\kappa_l - \kappa_i > 1$  and  $\frac{\kappa_l}{\kappa_1} \leq \frac{1}{100}$ .

Using the condition  $|\kappa_i| < \delta\kappa_1$ , we have,

$$\kappa_l - \kappa_i \leq (\delta + \frac{1}{100})\kappa_1.$$

Then, we have,

$$\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|i) \geq e^{\kappa_l}\frac{1 - e^{-1}}{\kappa_l - \kappa_i}\sigma_{k-1}(\kappa|i) \geq \frac{1 - e^{-1}}{\frac{1}{100} + \delta}\frac{e^{\kappa_l}}{\kappa_1}\sigma_{k-1}(\kappa|i).$$

Now, choosing  $\delta$  sufficient small, we get,

$$\frac{1 - e^{-1}}{\frac{1}{100} + \delta} \geq 1.$$

Then insert the above two inequalities into (3.2), we get (3.1).

Case (iv):  $\kappa_l - \kappa_i > 1$  and  $\frac{\kappa_l}{\kappa_1} > \frac{1}{100}$ .

In this case, (3.1) can be rewritten,

$$(3.3) \quad (1 + \varepsilon_T)e^{\kappa_l}\sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T)\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}(\kappa_i\sigma_{k-2}(\kappa|il) + \sigma_{k-1}(\kappa|il)) \\ \geq \frac{e^{\kappa_l}}{\kappa_1}(\kappa_l\sigma_{k-2}(\kappa|il) + \sigma_{k-1}(\kappa|il)).$$

If  $\sigma_{k-1}(\kappa|il) \leq 0$ , (3.3) is clearly true. Thus, we can assume  $\sigma_{k-1}(\kappa|il) > 0$ . Obviously, we have,

$$e^{\kappa_l} \sigma_{k-2}(\kappa|il) \geq \frac{e^{\kappa_l}}{\kappa_1} \kappa_l \sigma_{k-2}(\kappa|il).$$

To prove (3.3), we only need to show the following two inequalities,

$$(3.4) \quad (1 + \varepsilon_T) \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{k-1}(\kappa|il) \geq \frac{e^{\kappa_l}}{\kappa_1} \sigma_{k-1}(\kappa|il),$$

and

$$(3.5) \quad \varepsilon_T e^{\kappa_l} \sigma_{k-2}(\kappa|il) + (1 + \varepsilon_T) \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \kappa_i \sigma_{k-2}(\kappa|il) \geq 0.$$

To obtain (3.4), since  $\sigma_{k-1}(\kappa|il) > 0$ , we can take off it in both sides. Hence, we only need,

$$\varepsilon_T \kappa_1 e^{\kappa_l} + \kappa_i e^{\kappa_l} - (1 + \varepsilon_T) \kappa_1 e^{\kappa_i} \geq 0.$$

Using  $|\kappa_i| < \delta \kappa_1$ , we need,

$$(\varepsilon_T - \delta) \kappa_1 e^{\kappa_l} - (1 + \varepsilon_T) \kappa_1 e^{\kappa_i} \geq 0,$$

which implies the following requirement,

$$\kappa_l - \kappa_i \geq \log\left(\frac{1 + \varepsilon_T}{\varepsilon_T - \delta}\right).$$

Since  $|\kappa_i| < \delta \kappa_1$ ,  $\kappa_l > \frac{1}{100} \kappa_1$ , the above requirement can be satisfied, if

$$\left(\frac{1}{100} - \delta\right) \kappa_1 \geq \log\left(\frac{1 + \varepsilon_T}{\varepsilon_T - \delta}\right).$$

Hence, taking sufficient large  $\kappa_1$ , we obtain the above inequality.

In order to get (3.5), we need,

$$\varepsilon_T + (1 + \varepsilon_T) \frac{1 - e^{\kappa_i - \kappa_l}}{\kappa_l - \kappa_i} \kappa_i \geq 0.$$

If  $\kappa_i \geq 0$ , it is clearly right. Hence, we only consider the case  $\kappa_i < 0$ . Then, we need to require,

$$(\kappa_l - \kappa_i) \varepsilon_T \geq -(1 + \varepsilon_T) \kappa_i,$$

which implies,

$$\kappa_l \varepsilon_T \geq -\kappa_i.$$

By our assumption,  $\kappa_l \geq \frac{1}{100} \kappa_1$ ,  $|\kappa_i| \leq \delta \kappa_1$ , we only need the constants  $\delta$  and  $\varepsilon_T$  to satisfy,

$$\frac{\varepsilon_T}{100} \geq \delta.$$

We complete our proof. □

Now we consider the global  $C^2$ -estimates for the curvature equation (1.1).

Set  $u(X) = \langle X, \nu(X) \rangle$ . By the assumption that  $M$  is starshaped with a  $C^1$  bound,  $u$  is bounded from below and above by two positive constants. At every point in the hypersurface  $M$ , choose a local coordinate frame  $\{\partial/(\partial x_1), \dots, \partial/(\partial x_{n+1})\}$  in  $\mathbb{R}^n$  such that the first  $n$  vectors are the local coordinates of the hypersurface and the last one is the unit outer normal vector. Denote  $\nu$  to be the outer normal vector. We let  $h_{ij}$  and  $u$  be the second fundamental form and the support function of the hypersurface  $M$  respectively. The following geometric formulas are well known (e.g., [19]).

$$(3.6) \quad h_{ij} = \langle \partial_i X, \partial_j \nu \rangle,$$

and

$$(3.7) \quad \begin{aligned} X_{ij} &= -h_{ij}\nu \quad (\text{Gauss formula}) \\ (\nu)_i &= h_{ij}\partial_j \quad (\text{Weigarten equation}) \\ h_{ijk} &= h_{ikj} \quad (\text{Codazzi formula}) \\ R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \quad (\text{Gauss equation}), \end{aligned}$$

where  $R_{ijkl}$  is the  $(4,0)$ -Riemannian curvature tensor. We also have

$$(3.8) \quad \begin{aligned} h_{ijkl} &= h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} \\ &= h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \end{aligned}$$

For function  $u$ , we consider the following test function which appear firstly in [21],

$$\phi = \log \log P - N \ln u.$$

Here the function  $P$  is defined by

$$P = \sum_l e^{\kappa_l}.$$

We may assume that the maximum of  $\phi$  is achieved at some point  $X_0 \in M$ . After rotating the coordinates, we may assume the matrix  $(h_{ij})$  is diagonal at the point, and we can further assume that  $h_{11} \geq h_{22} \geq \dots \geq h_{nn}$ . Denote  $\kappa_i = h_{ii}$ .

Differentiate the function twice at  $X_0$ , we have,

$$(3.9) \quad \phi_i = \frac{P_i}{P \log P} - N \frac{h_{ii} \langle X, \partial_i \rangle}{u} = 0,$$

and,

$$\begin{aligned}
& \phi_{ii} \\
&= \frac{P_{ii}}{P \log P} - \frac{P_i^2}{P^2 \log P} - \frac{P_i^2}{(P \log P)^2} - \frac{N}{u} \sum_l h_{il,i} \langle \partial_l, X \rangle - \frac{N h_{ii}}{u} \\
& \quad + N h_{ii}^2 + N \frac{h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{\alpha \neq \beta} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \\
& \quad - \frac{N \sum_l h_{iil} \langle \partial_l, X \rangle}{u} - \frac{N h_{ii}}{u} + N h_{ii}^2 + N \frac{h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} h_{ii, ll} + \sum_l e^{\kappa_l} (h_{il}^2 - h_{ii} h_{ll}) h_{ii} + \sum_l e^{\kappa_l} (h_{ii} h_{ll} - h_{il}^2) h_{ll} \right. \\
& \quad \left. + \sum_l e^{\kappa_l} h_{lli}^2 + \sum_{\alpha \neq \beta} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \\
& \quad - \frac{N \sum_l h_{iil} \langle \partial_l, X \rangle}{u} - \frac{N h_{ii}}{u} + N h_{ii}^2 + N \frac{h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2}
\end{aligned}$$

Contract with  $\sigma_{n-1}^{ii}$ ,

$$\begin{aligned}
(3.10) \quad & \sigma_{n-1}^{ii} \phi_{ii} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \sigma_{n-1}^{ii} h_{ii, ll} + (n-1) f \sum_l e^{\kappa_l} h_{ll}^2 - \sigma_{n-1}^{ii} h_{ii}^2 \sum_l e^{\kappa_l} h_{ll} \right. \\
& \quad \left. + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{lli}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \right] \\
& \quad - \frac{N \sum_l \sigma_{n-1}^{ii} h_{iil} \langle \partial_l, X \rangle}{u} - \frac{N(n-1) f}{u} + N \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2}.
\end{aligned}$$

At  $x_0$ , differentiate equation (1.1) twice, we have,

$$(3.11) \quad \sigma_{n-1}^{ii} h_{iik} = d_X f(\partial_k) + h_{kk} d_\nu f(\partial_k),$$

and

$$(3.12) \quad \sigma_{n-1}^{ii} h_{iikk} + \sigma_{n-1}^{pq, rs} h_{pqk} h_{rsk} \geq -C - C h_{11}^2 + \sum_l h_{lkk} d_\nu f(\partial_l),$$

where  $C$  is some constant under control.



Insert (3.12) into (3.10),

$$\begin{aligned}
(3.13) \quad & \sigma_{n-1}^{ii} \phi_{ii} \\
& \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} (-C - Ch_{11}^2 - \sigma_{n-1}^{pq,rs} h_{pql} h_{rsl}) + \sum_l e^{\kappa_k} h_{lkk} d_\nu f(\partial_l) \right. \\
& \quad + (n-1)f \sum_l e^{\kappa_l} h_{ll}^2 - \sigma_{n-1}^{ii} h_{ii}^2 \sum_l e^{\kappa_l} h_{ll} + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{lli}^2 \\
& \quad + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \Big] \\
& \quad - \frac{N \sum_l \sigma_{n-1}^{ii} h_{iil} \langle \partial_l, X \rangle}{u} - \frac{N(n-1)f}{u} + N \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2}.
\end{aligned}$$

By (3.9) and (3.11), we have,

$$\begin{aligned}
(3.14) \quad & \sum_k d_\nu f(\partial_k) \frac{\sum_l e^{\kappa_l} h_{llk}}{P \log P} - \frac{N}{u} \sum_k \sigma_{n-1}^{ii} h_{iik} \langle \partial_k, X \rangle \\
& = -\frac{N}{u} \sum_k d_X f(\partial_k) \langle X, \partial_k \rangle.
\end{aligned}$$

Denote

$$\begin{aligned}
A_i &= e^{\kappa_i} (K(\sigma_{n-1})_i^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppi} h_{qqi}), \quad B_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} h_{lli}^2, \\
C_i &= \sigma_{n-1}^{ii} \sum_l e^{\kappa_l} h_{lli}^2, \quad D_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ll} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} h_{lli}^2, \quad E_i = \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2.
\end{aligned}$$

Using

$$-\sum_l \sigma_{n-1}^{pq,rs} h_{pql} h_{rsl} = \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{qq l},$$

and (3.13), for any  $K > 1$ , we have,

$$\begin{aligned}
(3.15) \quad & \sigma_{n-1}^{ii} \phi_{ii} \\
& \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} (K(\sigma_{n-1})_l^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{qq l} + \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2) \right. \\
& \quad + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{lli}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2 \\
& \quad \left. - CP - CKPh_{11}^2 \right] + (N-1) \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2} \\
& \geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i) \\
& \quad + (N-1) \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2} - \frac{C + CKh_{11}^2}{\log P}.
\end{aligned}$$

**Lemma 12.** *There exists a constant  $\delta < \frac{1}{2}$  such that, if  $|\kappa_i| \leq \delta\kappa_1$ , we have,*

$$A_i + B_i + C_i + D_i - E_i \geq 0,$$

for sufficient large  $K$  and  $\kappa_1$ .

*Proof.* Firstly, using Lemma 6, we have  $A_i > 0$ , for sufficient large constant  $K$ . By Cauchy-Schwarz inequality, we have,

$$\begin{aligned} (3.16) \quad P_i^2 &= e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iii} h_{lli} + \left( \sum_{l \neq i} e^{\kappa_l} h_{lli} \right)^2 \\ &\leq e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iii} h_{lli} + (P - e^{\kappa_i}) \sum_{l \neq i} e^{\kappa_l} h_{lli}^2. \end{aligned}$$

Using (3.16), we have,

$$\begin{aligned} (3.17) \quad B_i + C_i + D_i - E_i &\geq 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll, ii} h_{lli}^2 + 2 \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ii} h_{lli}^2 \\ &\quad + \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ &\quad - \frac{1 + \log P}{P \log P} e^{2\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - 2 \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_{n-1}^{ii} h_{iii} h_{lli}. \end{aligned}$$

Using Lemma 11, there exists a constant  $\delta < \frac{1}{2}$ , such that,

$$(3.18) \quad \frac{3}{2} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll, ii} h_{lli}^2 + \frac{3}{2} \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ii} h_{lli}^2 \geq 0.$$

On the other hand, we see that,

$$(3.19) \quad \sum_{l \neq i, 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 - 2 \sum_{l \neq i, 1} e^{\kappa_i + \kappa_l} \sigma_{n-1}^{ii} h_{iii} h_{lli} \geq - \sum_{l \neq i, 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2.$$

Then, using the above two inequalities, (3.17) becomes,

$$\begin{aligned} (3.20) \quad B_i + C_i + D_i - E_i &\geq \frac{1 + \log P}{P \log P} e^{\kappa_1 + \kappa_i} \sigma_{n-1}^{ii} h_{11i}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ &\quad - \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - 2 \frac{1 + \log P}{P \log P} e^{\kappa_i + \kappa_1} \sigma_{n-1}^{ii} h_{iii} h_{11i} \\ &\quad + \frac{1}{2} e^{\kappa_1} \sigma_{n-1}^{11, ii} h_{11i}^2 + \frac{1}{2} \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{11} h_{11i}^2. \end{aligned}$$

Directly calculation shows that,

$$\begin{aligned} e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 &\geq \left( \frac{e^{\kappa_1}}{P} - \frac{1}{\log P} \right) e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ &\geq \frac{1}{n+1} e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \end{aligned}$$

and

$$-2 \frac{1 + \log P}{P \log P} e^{\kappa_i + \kappa_1} \sigma_{n-1}^{ii} |h_{iii} h_{11i}| \geq -\frac{3}{P} e^{\kappa_i + \kappa_1} \sigma_{n-1}^{ii} |h_{iii} h_{11i}| \geq -3 e^{\kappa_i} \sigma_{n-1}^{ii} |h_{iii} h_{11i}|,$$

hold for sufficient large  $\kappa_1$ . We let  $l = 1, k = n - 1$  in (3.9), we have,

$$(3.21) \quad e^{\kappa_1} \sigma_{n-1}^{11, ii} h_{11i}^2 + \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{11} h_{11i}^2 = e^{\kappa_i} \sigma_{n-1}^{11, ii} h_{11i}^2 + \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{ii} h_{11i}^2.$$

By Taylor expansion, we also have,

$$(3.22) \quad \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{ii} h_{11i}^2 = e^{\kappa_i} \sum_{m \geq 1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} \sigma_{n-1}^{ii} h_{11i}^2.$$

Combining the previous four formulas and using (3.20), we obtain,

$$B_i + C_i + D_i - E_i \geq e^{\kappa_i} \sigma_{n-1}^{ii} \left[ \frac{1}{n+1} h_{iii}^2 - 3 |h_{iii} h_{11i}| + \frac{1}{2} \sum_{m \geq 1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} h_{11i}^2 \right] \geq 0,$$

for sufficient large  $\kappa_1$ .  $\square$

In  $\Gamma_{n-1}$  cone, it is well known that the only possible negative eigenvalue is the smallest one. Since we have assumed that  $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n$ , the possible non positive eigenvalue is  $\kappa_n$ . Hence, we can state the following little Lemma.

**Lemma 13.** *In  $\Gamma_{n-1}$  cone, if  $\kappa_n \leq 0$ , we have,*

$$-\kappa_n \leq \frac{\kappa_1}{n-1}.$$

*Proof.* It is easy to see that,

$$\sigma_{n-1}(\kappa|n) = \kappa_1 \cdots \kappa_{n-1}, \text{ and } \sigma_{n-2}(\kappa|1n) = \kappa_2 \cdots \kappa_{n-1}.$$

We assume that  $\lambda = -\kappa_n / \kappa_1$ . Then we have,

$$\begin{aligned} \kappa_1 \cdots \kappa_{n-1} &= \sigma_{n-1} - \kappa_n \sigma_{n-2}(\kappa|n) \\ &> -\kappa_n \sigma_{n-2}(\kappa|n) = \lambda \kappa_1 \sigma_{n-2}(\kappa|n) \\ &= \lambda \kappa_1^2 \sigma_{n-3}(\kappa|n1) + \lambda \kappa_1 \sigma_{n-2}(\kappa|1n). \end{aligned}$$

Hence, we get,

$$(1 - \lambda) \kappa_2 \cdots \kappa_{n-1} > \lambda \kappa_1 \sigma_{n-3}(\kappa|n1) \geq (n-2) \lambda \kappa_2 \cdots \kappa_{n-1},$$

which implies  $\lambda < \frac{1}{n-1}$ .  $\square$

**Lemma 14.** For the chosen constant  $\delta$  in Lemma 12, if  $\kappa_i \geq \delta\kappa_1$  and  $n \geq 3$ , we have,

$$A_i + B_i + C_i + D_i - E_i \geq 0,$$

for sufficient large  $K$  and  $\kappa_1$ .

*Proof.* Using (3.17), we have,

$$\begin{aligned}
(3.23) \quad & A_i + B_i + C_i + D_i - E_i \\
& \geq e^{\kappa_i} (K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq} h_{ppi} h_{qqi}) + 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll,ii} h_{lli}^2 \\
& \quad - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ii} h_{lli}^2 + \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 \\
& \quad + 2 \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{lli}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - \frac{1 + \log P}{P \log P} e^{2\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\
& \quad - 2 \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_{n-1}^{ii} h_{iii} h_{lli}.
\end{aligned}$$

We claim that the following inequality holds for sufficient large  $\kappa_1$ ,

$$(3.24) \quad e^{\kappa_i} (K(\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq} h_{ppi} h_{qqi}) + 2 \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{lli}^2 \geq \frac{1}{\log P} e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2.$$

In view of Proposition 8, we need to prove that, for given arbitrary small constant  $\epsilon$ , if  $\kappa_1$  is sufficient large, we have,

$$(3.25) \quad 2 \frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} \kappa_1 \geq 2 \frac{n-1}{n} - \epsilon,$$

for all  $l \neq i$ . We divide into three cases to discuss.

Case (i):  $\kappa_l \geq \kappa_i$ . In this case, we obviously have,

$$\frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} = \frac{e^{\kappa_l - \kappa_i} - 1}{\kappa_l - \kappa_i} \geq 1.$$

It is easy to get (3.25) for sufficient large  $\kappa_1$ .

Case (ii):  $\kappa_i - \kappa_l \geq C_0$  where we take,

$$C_0 \geq \log \frac{2(n-1)}{n\epsilon}.$$

Then, we have,

$$2 \frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} \kappa_1 \geq \frac{2\kappa_1}{\kappa_i - \kappa_l} (1 - e^{-C_0}).$$

Since  $0 < \kappa_i \leq \kappa_1$ , if  $\kappa_l \geq 0$ , it is easy to see,

$$\frac{2\kappa_1}{\kappa_i - \kappa_l} \geq 2 \frac{n-1}{n}.$$

If  $\kappa_l < 0$ , in  $\Gamma_{n-1}$ , we only have one negative eigenvalue, by Lemma 13, we have,

$$\frac{2\kappa_1}{\kappa_i - \kappa_l} \geq 2\frac{n-1}{n}.$$

Combining the previous four inequalities and  $n \geq 3$ , we have (3.25).

Case (iii):  $0 < \kappa_i - \kappa_l \leq C_0$  where  $C_0$  is defined in the previous case. In this case, using mean value theorem, we have,

$$\frac{1 - e^{\kappa_l - \kappa_i}}{\kappa_i - \kappa_l} = \frac{1}{e^{\kappa_i}} \frac{e^{\kappa_i} - e^{\kappa_l}}{\kappa_i - \kappa_l} = \frac{e^\xi}{e^{\kappa_i}} \geq \frac{e^{\kappa_l}}{e^{\kappa_i}} \geq e^{-C_{\varepsilon_N}}.$$

Here  $\xi$  is the mean value of  $\kappa_i$  and  $\kappa_l$ . Since  $\kappa_1$  is sufficient large, this yields (3.25). In a word, (3.24) holds for any case.

Note that, in cone  $\Gamma_{n-1}$ ,

$$\begin{aligned} 2\kappa_1\sigma_{n-3}(\kappa|il) - \sigma_{n-2}(\kappa|i) &= 2\kappa_1\sigma_{n-3}(\kappa|il) - \kappa_l\sigma_{n-3}(\kappa|il) - \sigma_{n-2}(\kappa|il) \\ &\geq \kappa_1\sigma_{n-3}(\kappa|il) - \sigma_{n-2}(\kappa|il) \\ &= \kappa_1^2\sigma_{n-4}(\kappa|il1) + \kappa_1\sigma_{n-3}(\kappa|il1) - \sigma_{n-2}(\kappa|il) \\ &= \kappa_1^2\sigma_{n-4}(\kappa|il1) > 0, \end{aligned}$$

is true for all  $i, l$ . It implies,

$$2\kappa_1\sigma_{n-1}^{ll,ii} \geq \sigma_{n-1}^{ii}.$$

Using the above inequality, we have,

$$(3.26) \quad 2 \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll,ii} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ii} h_{lli}^2 \geq 0.$$

On the other hand, we have,

$$\begin{aligned} (3.27) \quad & \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{lli}^2 - 2 \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_{n-1}^{ii} h_{iii} h_{lli} \\ & \geq - \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2. \end{aligned}$$

Inserting (3.24), (3.26) and (3.27) into (3.23), we obtain,

$$\begin{aligned} & A_i + B_i + C_i + D_i - E_i \\ & \geq \frac{1}{\log P} e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - \frac{1 + \log P}{P \log P} e^{2\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ & \quad - \frac{1 + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ & = 0. \end{aligned}$$

□

For the negative part, we have the following estimate.

**Lemma 15.** *If  $-\kappa_i \geq \delta\kappa_1$  and  $n \geq 3$ , then we also have,*

$$A_i + B_i + C_i + D_i - E_i \geq 0,$$

*for sufficient large  $K$  and  $\kappa_1$ .*

*Proof.* Firstly, for sufficient large constant  $K$ , by Lemma 6, we have  $A_i > 0$ . In this case, the only possible negative eigenvalue is  $\kappa_n$ . By Lemma 13, we know that  $-\kappa_i < \frac{1}{n-1}\kappa_1$ . Then using the similar argument of the inequality (3.25) in the previous Lemma, we have,

$$\frac{5\kappa_1}{3} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \geq \frac{5}{6} (2\frac{n-1}{n} - \epsilon) e^{\kappa_l}.$$

Since  $n \geq 3$ , the coefficient of the right hand side in the above inequality is bigger than 1 for sufficient small  $\epsilon$ . Hence, using Lemma 11, we have,

$$(3.28) \quad \frac{5}{3} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ll,ii} h_{lli}^2 + \frac{5}{3} \sum_{l \neq i} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_{n-1}^{ll} h_{lli}^2 - \frac{1}{\log P} \sum_{l \neq i} e^{\kappa_l} \sigma_{n-1}^{ii} h_{lli}^2 \geq 0.$$

Using (3.19), (3.28) and (3.17), we obtain,

$$\begin{aligned} & B_i + C_i + D_i - E_i \\ \geq & \frac{1 + \log P}{P \log P} e^{\kappa_1 + \kappa_i} \sigma_{n-1}^{ii} h_{11i}^2 + e^{\kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 \\ & - \frac{1 + \log P}{P \log P} \sum_{l \neq 1} e^{\kappa_l + \kappa_i} \sigma_{n-1}^{ii} h_{iii}^2 - 2 \frac{1 + \log P}{P \log P} e^{\kappa_i + \kappa_1} \sigma_{n-1}^{ii} h_{iii} h_{11i} \\ & + \frac{1}{3} e^{\kappa_1} \sigma_{n-1}^{11,ii} h_{11i}^2 + \frac{1}{3} \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_{n-1}^{11} h_{11i}^2. \end{aligned}$$

The last expression is similar to (3.20). Thus, using similar argument in Lemma 12, it is nonnegative.  $\square$

Now, we are in the position to prove our main theorem.

**Proof of Theorem 2:** For  $n \geq 3$ , using Lemma 12, Lemma 14 and Lemma 15 in (3.15), we obtain,

$$\begin{aligned} 0 & \geq \sigma_{n-1}^{ii} \phi_{ii} \\ & \geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i) \\ & \quad + (N-1) \sigma_{n-1}^{ii} h_{ii}^2 + N \frac{\sigma_{n-1}^{ii} h_{ii}^2 \langle X, \partial_i \rangle^2}{u^2} - \frac{C + CK h_{11}^2}{\log P} \\ & \geq (N-1) c_0 h_{11} - \frac{C + CK h_{11}^2}{\log P}. \end{aligned}$$

Here we have used

$$\sigma_{n-1}^{11} h_{11} \geq c_0.$$

Choosing sufficient large  $N$ , we get an upper bound of  $h_{11}$ .

For  $n = 2$ , the equation is a quasi linear elliptic equation. The  $C^2$  estimate is well known.

#### 4. SOME APPLICATION

Let's gives some applications. The first is to prove existence result, Theorem 4.

**Proof of Theorem 4:** We use continuity method to solve the existence result. For  $0 \leq t \leq 1$ , according to [10], we consider the family of functions,

$$f^t(X, \nu) = tf(X, \nu) + (1 - t)C_n^2[\frac{1}{|X|^k} + \varepsilon(\frac{1}{|X|^k} - 1)],$$

where  $\varepsilon$  is sufficient small constant satisfying

$$0 < f_0 \leq \min_{r_1 \leq \rho \leq r_2} (\frac{1}{\rho^k} + \varepsilon(\frac{1}{\rho^k} - 1)),$$

and  $f_0$  is some positive constant. The  $C^0$  and  $C^1$  estimates is same to the proof in [21]. For  $n \geq 3$ , the  $C^2$  estimate comes from Theorem 2. The openness comes from [10]. By continuity method and Evans-Krylov theory, we obtain Theorem 4. We complete our proof.

The proof of the Corollary 3 is similar to Theorem 2. Using the Corollary and the boundary estimates obtained in [17], we have the following existence result for Dirichlet problem.

**Theorem 16.** *Suppose  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary. Suppose  $f(p, u, x) \in C^2(\mathbb{R}^n \times \mathbb{R} \times \bar{\Omega})$  is a positive function with  $f_u \geq 0$ . Suppose there is a subsolution  $\underline{u} \in C^3(\bar{\Omega})$  satisfying*

$$(4.1) \quad \begin{cases} \sigma_{n-1}[D^2 \underline{u}] & \geq f(x, \underline{u}, D\underline{u}), \\ \underline{u}|_{\partial\Omega} & = \varphi. \end{cases}$$

*then the Dirichlet problem (1.3) has a unique  $C^{3,\alpha}$  solution  $u$  for any  $0 < \alpha < 1$ .*

Then, we consider the prescribed curvature problem for spacelike graph hypersurface in Minkowski space.

We present some setting of that problem. If function  $u$  is the description function and hypersurface  $M = \text{graph } u$ .  $u$  is defined in some bounded domain  $\Omega \subset \mathbb{R}^n$ . The Minkowski space  $\mathbb{R}^{n,1}$  is defined by the following metric,

$$ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

Since  $M$  is space like, in [7], the uniformly  $C^1$  bound has been obtained for equation (1.6). Namely, there is some constant  $\theta$ , such that,

$$\sup_{\bar{\Omega}} |Du| \leq \theta < 1.$$

The induce metric on  $M$  is,

$$g_{ij} = \delta_{ij} - D_i u D_j u, \quad 1 \leq i, j \leq n.$$

The second fundamental form is,

$$h_{ij} = \frac{D_{ij}u}{\sqrt{1 - |Du|^2}}.$$

We still denote the principal curvature of  $M$  by  $\kappa_1, \dots, \kappa_n$ . We also define the second fundamental form,

$$(4.2) \quad h_{ij} = \langle \partial_i X, \partial_j \nu \rangle,$$

Here  $\langle, \rangle$  is the Minkowski inner product defined by metric  $ds^2$  in the above. Then, for space like hypersurface, we have different Gauss formula and Gauss equation,

$$(4.3) \quad \begin{aligned} X_{ij} &= h_{ij}\nu \quad (\text{Gauss formula}) \\ R_{ijkl} &= -(h_{ik}h_{jl} - h_{il}h_{jk}) \quad (\text{Gauss equation}), \end{aligned}$$

where  $R_{ijkl}$  is the  $(4,0)$ -Riemannian curvature tensor. Hence, the communication formula also change a little bit,

$$(4.4) \quad \begin{aligned} h_{ijkl} &= h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} \\ &= h_{klij} - (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} - (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \end{aligned}$$

Now let's give the proof of Theorem 5.

**Proof of Theorem 5:**  $C^0$  estimate comes from comparison principal. We also have the  $C^1$  estimate. For  $C^2$  estimates on the boundary, using the sub solution and the  $C^2$  boundary estimate argument [17], we can obtain it. For the interior, we use the similar trick in section 3. Hence, for function  $u$ , we consider the following test function,

$$\phi = \log \log P + \frac{N}{2}|Du|^2.$$

where function  $P$  is also defined by

$$P = \sum_l e^{\kappa_l}.$$

Suppose that  $M$  achieve its maximum value in  $\Omega$  at some point  $x_0$ . We can assume that matrix  $(u_{ij})$  is diagonal by rotating the coordinate, and  $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n$ . Hence, at  $x_0$ , differentiating  $\phi$  twice, we have

$$(4.5) \quad \phi_i = \frac{P_i}{P \log P} + Nu_i u_{ii} = 0,$$

and,

$$(4.6) \quad \phi_{ii} = \frac{P_{ii}}{P \log P} - \frac{(1 + \log P)P_i^2}{(P \log P)^2} + \sum_s Nu_s u_{sii} + Nu_{ii}^2.$$



Similar to the calculation (3.9) and (3.10), we have,

$$\begin{aligned}
(4.7) \quad & \sigma_{n-1}^{ii} \phi_{ii} \\
&= \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \sigma_{n-1}^{ii} h_{ii,ll} - (n-1)f \sum_l e^{\kappa_l} h_{ll}^2 + \sigma_{n-1}^{ii} h_{ii}^2 \sum_l e^{\kappa_l} h_{ll} \right. \\
&\quad + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{ll}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \Big] \\
&\quad + \sum_s N u_s \sigma_{n-1}^{ii} u_{sii} + \sigma_{n-1}^{ii} N u_{ii}^2.
\end{aligned}$$

At  $x_0$ , differentiating equation (1.6) twice, we have,

$$(4.8) \quad \sigma_{n-1}^{ii} h_{iij} = f_j + f_u u_j + f_{p_j} u_{jj},$$

and

$$(4.9) \quad \sigma_{n-1}^{ii} h_{iijj} + \sigma_{n-1}^{pq,rs} h_{pqj} h_{rsj} \geq -C - C u_{jj}^2 + \sum_s f_{p_s} u_{sjj}.$$

Inserting (4.9) into (4.7), we have

$$\begin{aligned}
(4.10) \quad & \sigma_{n-1}^{ii} \phi_{ii} \\
&\geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} (K(\sigma_{n-1})_l^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{ppl} h_{qq l} + \sum_{p \neq q} \sigma_{n-1}^{pp,qq} h_{pql}^2) \right. \\
&\quad + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} h_{ll}^2 + \sum_{\alpha \neq \beta} \sigma_{n-1}^{ii} \frac{e^{\kappa_\alpha} - e^{\kappa_\beta}}{\kappa_\alpha - \kappa_\beta} h_{\alpha\beta i}^2 - \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2 \\
&\quad \left. - CP - CK P h_{11}^2 \right] + N \sigma_{n-1}^{ii} u_{ii}^2 \\
&\geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i) + N \sigma_{n-1}^{ii} h_{ii}^2 (1 - |Du|^2) \\
&\quad - \frac{C + CK \kappa_1^2}{\log P}.
\end{aligned}$$

Here, the definition of  $A_i, B_i, C_i, D_i, E_i$  is same meaning as the previous section. Thus, since  $\theta$  is a constant smaller than 1, we obtain the uniformly bound of  $h_{11}$ . The openness is standard. Using the continuity method and Evans-Krylov theory, we obtain our theorem.

*Acknowledgement:* The authors wish to thank Professor Pengfei Guan for his valuable suggestions and comments. They also thank the Shanghai Centre for Mathematical Sciences for their partial support. The first author would like to thank Fudan University for their support and hospitality.

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